DÚ 10 - ŘEŠENÍ

URČITÝ INTEGRÁL

PLOCHY OBRAZCŮ

1) $\displaystyle\int_{\frac{3\pi}{4}}^{\pi} (4x+\pi)\sin x \, dx =$

$$= \left| \begin{array}{ll} u' = \sin x & v = 4x+\pi \\ u = -\cos x & v' = 4 \end{array} \right|_{\frac{3\pi}{4}}^{\pi} = \Big[-\cos x\,(4x+\pi)\Big]_{\frac{3\pi}{4}}^{\pi} - \int_{\frac{3\pi}{4}}^{\pi} -\cos x \cdot 4 \, dx =$$

$$= \Big[-\cos x\,(4x+\pi) + 4\sin x\Big]_{\frac{3\pi}{4}}^{\pi} = \Big[-\cos\pi\,(4\pi+\pi) + 4\sin\pi\Big] - \Big[-\cos\left(\tfrac{3\pi}{4}\right)\left(4\,\tfrac{3\pi}{4}+\pi\right) + 4\sin\tfrac{3\pi}{4}\Big] =$$

$$= \Big[-(-1)\cdot 5\pi + 4\cdot 0\Big] - \Big[-\left(-\tfrac{\sqrt{2}}{2}\right)\cdot 4\pi + 4\cdot\tfrac{\sqrt{2}}{2}\Big] = \Big[5\pi\Big] - \Big[\tfrac{\sqrt{2}}{2}\cdot 4\pi + 2\sqrt{2}\Big] =$$

$$= \boxed{5\pi - 2\sqrt{2}\pi - 2\sqrt{2}}$$

2) $\displaystyle\int_{-\pi}^{2\pi} \sin \frac{x+3\pi}{6} \, dx =$

$$= \left| \begin{array}{l} t = \frac{x+3\pi}{6} \\ dt = \frac{1}{6}\,dx \\ dx = 6\,dt \end{array} \quad \begin{array}{c|c|c} x & 2\pi & -\pi \\ \hline t & \frac{5\pi}{6} & \frac{\pi}{3} \end{array} \right| = \int_{\frac{\pi}{3}}^{\frac{5\pi}{6}} \sin t \cdot 6 \, dt = \Big[-6\cos t\Big]_{\frac{\pi}{3}}^{\frac{5\pi}{6}} = -6\cos\frac{5\pi}{6} + 6\cos\frac{\pi}{3} =$$

$$= -6\cdot\frac{-\sqrt{3}}{2} + 6\cdot\frac{1}{2} = \boxed{3\sqrt{3}+3}$$

3) $\displaystyle\int_{-1}^{0} e^{-x^3}\cdot x^2 \, dx = \left| \begin{array}{l} t = -x^3 \\ dt = -3x^2\,dx \\ dx = \frac{dt}{-3x^2} \end{array} \quad \begin{array}{c|c|c} x & 0 & -1 \\ \hline t & 0 & 1 \end{array} \right| = \int_{1}^{0} e^{t}\cdot \cancel{x^2}\cdot\frac{dt}{-3\cancel{x^2}} = -\frac{1}{3}\int_{1}^{0} e^{t}\,dt =$

$$= -\frac{1}{3}\Big[e^{t}\Big]_{1}^{0} = -\frac{1}{3}\left(e^{0}-e^{1}\right) = -\frac{e^{0}-e^{1}}{3} = -\frac{1-e}{3} = \boxed{\frac{e-1}{3}}$$

4) $\displaystyle\int_1^e \frac{\ln^3 x}{x}\,dx =$

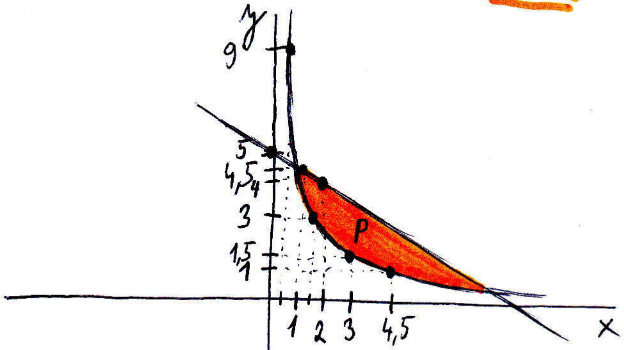
$$= \left| \begin{array}{l} t = \ln x \\ dt = \frac{1}{x}dx \quad \begin{array}{c|c|c} x & e & 1 \\ \hline t & 1 & 0 \end{array} \\ dx = x\,dt \end{array} \right| = \int_0^1 \frac{t^3}{\cancel{x}} \cdot \cancel{x}\,dt = \int_0^1 t^3\,dt = \left[\frac{t^4}{4}\right]_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \boxed{\frac{1}{4}}$$

5) $2x \cdot y = 9 \qquad x + 2y = 10$

$y = \frac{9}{2x} \qquad y = 5 - \frac{x}{2}$

x	1	0,5	1,5	3	4,5
y	4,5	9	3	1,5	1

x	0	2
y	5	4



MEZE: $\frac{9}{2x} = 5 - \frac{x}{2} \quad /\cdot 2x$

$9 = 10x - x^2$

$x^2 - 10x + 9 = 0$

$(x-1)(x-9) = 0$

$\downarrow 1 \qquad \downarrow 9$

$$P = \int_1^9 \left(5 - \frac{x}{2} - \frac{9}{2x}\right)dx = \int_1^9 5\,dx - \int_1^9 \frac{x}{2}\,dx - \int_1^9 \frac{9}{2x}\,dx =$$

$$= 5\int_1^9 dx - \frac{1}{2}\int_1^9 x\,dx - \frac{9}{2}\int_1^9 \frac{1}{x}\,dx = \left[5x - \frac{1}{2}\cdot\frac{x^2}{2} - \frac{9}{2}\ln|x|\right]_1^9 =$$

$$= \left[5\cdot 9 - \frac{81}{4} - \frac{9}{2}\ln 9\right] - \left[5 - \frac{1}{4} - \frac{9}{2}\ln 1\right] =$$

$$= 45 - \frac{81}{4} - \frac{9}{2}\ln 9 - 5 + \frac{1}{4} + 0 = 40 - \frac{80}{4} - \frac{9}{2}\ln 9 = 40 - 20 - \frac{9}{2}\ln 9 = \boxed{20 - \frac{9}{2}\ln 9}$$

6) $x - y = 1 \qquad y = 2x^2 - 5x - 1$

MEZE:

$y = x - 1$

x	0	1
y	-1	0

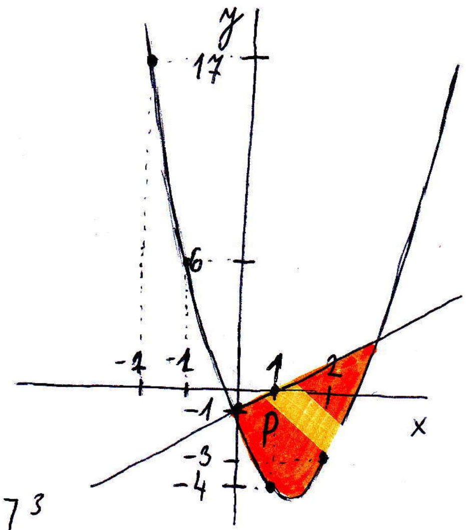
x	-2	-1	0	1	2
y	17	6	-1	-4	-3

$2x^2 - 5x - 1 = x - 1$

$2x^2 - 6x = 0$

$2x(x-3) = 0$

$\downarrow 0 \qquad \downarrow 3$

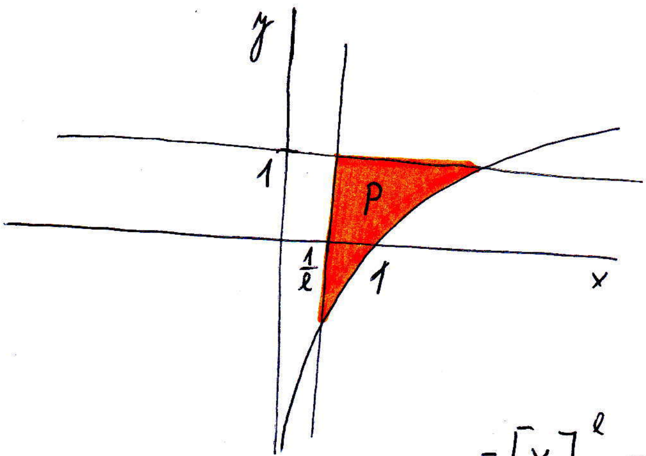


$$P = \int_0^3 [(x-1) - (2x^2 - 5x - 1)]\,dx = \int_0^3 (-2x^2 + 6x)\,dx =$$

$$= -2\int_0^3 x^2\,dx + 6\int_0^3 x\,dx = \left[-2\,\frac{x^3}{3} + \frac{6x^2}{2}\right]_0^3 = \left[-2\,\frac{x^3}{3} + 3x^2\right]_0^3 =$$

$$= -2\cdot\frac{3^3}{3} + 3\cdot 3^2 - 0 = -2\cdot 9 + 27 = -18 + 27 = \boxed{9}$$

7) $y = 1 \qquad y = \ln x \qquad x = \frac{1}{e}$ MEZE: $\frac{1}{e}$



$$\ln x = 1$$
$$\ln x = \ln e^{1}$$
$$x = e$$

$$P = \int_{\frac{1}{e}}^{e} (1 - \ln x)\,dx = \int_{\frac{1}{e}}^{e} 1\,dx - \int_{\frac{1}{e}}^{e} \ln x\,dx =$$

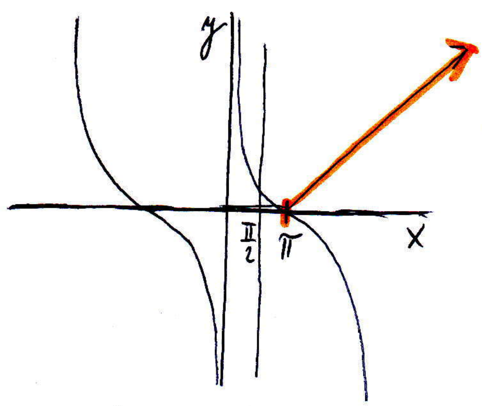
$$= [x]_{\frac{1}{e}}^{e} - \left| \begin{array}{ll} u' = 1 & v = \ln x \\ u = x & v' = \frac{1}{x} \end{array} \right|_{\frac{1}{e}}^{e} =$$

$$= [x]_{\frac{1}{e}}^{e} - \left\{ [x \cdot \ln x]_{\frac{1}{e}}^{e} - \int_{\frac{1}{e}}^{e} \cancel{x} \cdot \frac{1}{\cancel{x}}\,dx \right\} = \left[x - x \cdot \ln x + x \right]_{\frac{1}{e}}^{e} = \left[2x - x \cdot \ln x \right]_{\frac{1}{e}}^{e}$$

$$= [2 \cdot e - e \cdot \ln e] - \left[2 \cdot \frac{1}{e} - \frac{1}{e} \cdot \ln \frac{1}{e}\right] = 2e - e \cdot 1 - \frac{2}{e} + \frac{1}{e} \cdot \ln e^{-1} =$$

$$= e - \frac{2}{e} + \frac{1}{e} \cdot (-1) \cdot \ln e = e - \frac{2}{e} - \frac{1}{e} \cdot 1 = \boxed{e - \frac{3}{e}}$$

8) $\cotg \frac{x}{2} \qquad y = 0 \qquad x = \frac{\pi}{2}$



$$\cotg \frac{x}{2} = 0$$
$$\cotg \frac{x}{2} = \cotg \frac{\pi}{2}$$
$$\frac{x}{2} = \frac{\pi}{2}$$
$$x = \pi$$

$$P = \int_{\frac{\pi}{2}}^{\pi} \left(\cotg \frac{x}{2} - 0\right) dx = \int_{\frac{\pi}{2}}^{\pi} \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}}\,dx = \left| \begin{array}{l} t = \sin \frac{x}{2} \\ dt = \cos \frac{x}{2} \cdot \frac{1}{2}\,dx \\ dx = \frac{2\,dt}{\cos \frac{x}{2}} \\ \begin{array}{c|c|c} x & \pi & \frac{\pi}{2} \\ \hline t & 1 & \frac{\sqrt{2}}{2} \end{array} \end{array} \right| = \int_{\frac{\sqrt{2}}{2}}^{1} \frac{\cancel{\cos \frac{x}{2}}}{t} \cdot \frac{2\,dt}{\cancel{\cos \frac{x}{2}}} = 2 \int_{\frac{\sqrt{2}}{2}}^{1} \frac{dt}{t} =$$

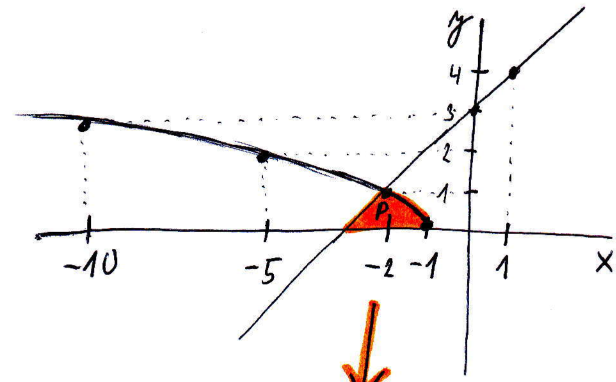
$$= [2 \ln t]_{\frac{\sqrt{2}}{2}}^{1} = 2 \cdot \ln 1 - 2 \ln \frac{\sqrt{2}}{2} = \boxed{-2 \ln \frac{\sqrt{2}}{2}} = \ln \left(\frac{\sqrt{2}}{2}\right)^{-2} = \ln \frac{(\sqrt{2})^{-2}}{2^{-2}} =$$

$$= \ln \frac{2^{-1}}{2^{-2}} = \ln \frac{2^{2}}{2} = \boxed{\ln 2}$$

9) $y = x+3 \qquad y = 0 \qquad y = \sqrt{-x-1}$

x	0	1
y	3	4

x	-10	-5	-2	-1
y	3	2	1	0



D: $x+3=0$

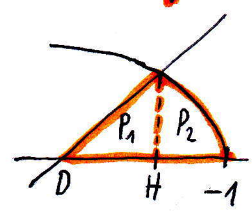
$x=-3$

H: $\sqrt{-x-1} = x+3 \quad /^2$

$-x-1 = x^2+6x+9$

$x^2+7x+10=0$

$(x+2)(x+5)=0 \quad \left.\begin{matrix} \nearrow x=-2 \\ \searrow x=-5 \end{matrix}\right\}$ z obrázku: $x=-2$



$$P_1 = \int_{-3}^{-2}(x+3-0)\,dx = \int_{-3}^{-2} x\,dx + \int_{-3}^{-2} 3\,dx = \left[\frac{x^2}{2}+3x\right]_{-3}^{-2} = \left[\frac{(-2)^2}{2}+3(-2)\right]-\left[\frac{(-3)^2}{2}+3(-3)\right] =$$

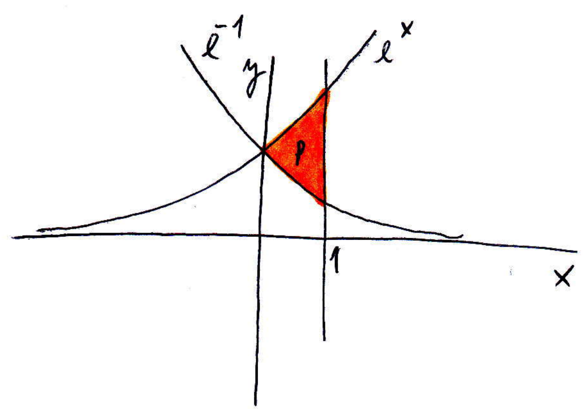
$$= \left(\frac{4}{2}-6\right)-\left(\frac{9}{2}-9\right) = 2-6-\frac{9}{2}+9 = 5-\frac{9}{2} = 5-4{,}5 = \underline{0{,}5}$$

$$P_2 = \int_{-2}^{-1}\left(\sqrt{-x-1}-0\right)dx = \int_{-2}^{-1}\sqrt{-x-1}\,dx = \left|\begin{matrix} t=-x-1 \\ dt=-1\,dx \\ dx=-dt \end{matrix} \quad \begin{array}{c|c|c} x & -1 & -2 \\ \hline t & 0 & 1 \end{array}\right| = \int_1^0 \sqrt{t}\,(-dt) = -\int_1^0 t^{\frac{1}{2}}dt =$$

$$= -\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^0 = -\left[\frac{2}{3}t^{\frac{3}{2}}\right]_1^0 = (-0)-\left(-\frac{2}{3}1^{\frac{3}{2}}\right) = \frac{2}{3}\cdot 1 = \underline{\frac{2}{3}}$$

$$\boxed{P = P_1+P_2 = \frac{1}{2}+\frac{2}{3} = \frac{3+4}{6} = \underline{\underline{\frac{7}{6}}}}$$

10) $y = e^x \qquad y = e^{-x} \qquad x = 1$



$$P = \int_0^1 \left(e^x - e^{-x}\right)dx = \int_0^1 e^x dx - \int_0^1 e^{-x}dx =$$

$$= \left[e^x\right]_0^1 - \left|\begin{matrix} t=-x \\ dt=-dx \\ dx=-dt \end{matrix} \quad \begin{array}{c|c|c} x & 1 & 0 \\ \hline t & -1 & 0 \end{array}\right| =$$

$$= \left[e^x\right]_0^1 - \int_0^{-1} e^t(-dt) = \left[e^x\right]_0^1 + \left[e^t\right]_0^{-1} = \left(e^1-e^0\right)+\left(e^{-1}-e^0\right) = e-1+\frac{1}{e}-1 =$$

$$= \boxed{\underline{\underline{e+\frac{1}{e}-2}}}$$